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Quantum Logic and Linear Logic

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1. INTRODUCTION

The purpose of this article is to clarify the relations between two logics. Namely, the so-called *quantum logic* and *linear logic*. Since Birkhoff and von Neumann (1936) wrote the paper, "The logic of quantum mechanics," quantum logic has been understood as the logic of the lattice of closed subspaces of a Hilbert space; it is well known that such lattices are *orthomodular* lattices. For some time the problem of dealing with these lattices was almost forgotten by mathematicians until Mulvey (1986) introduced the concept of a quantale. Various mathematicians have become interested in such kinds of lattices. These lattices are complete lattices with a noncommutative binary operation preserving all suprema. It was claimed that such kinds of lattices will provide a new foundation for the logic of quantum mechanics. Mulvey and Pelletier (1991) introduced the notions of a Gelfand quantale and a von Neumann quantale in order to explain the connections between quantales and the lattice of the closed subspaces of a Hilbert space. They used only half of the required information we need in order to define an orthomodular lattice. In fact, they used only the orthocomplement structure of the lattice of closed subspaces of a Hilbert space. We do not want to enter into a polemic on whether the so-called orthomodular law is important or not. We just mention the work by Loomis (1955) on dimension theory of operator algebras and the book by Kalmbach (1983), where the reader can find not only some basic results

This paper is dedicated to the memory of Dr. Alfredo Román Páez.

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concerning orthomodular lattices, but also a good bibliography on this subject. Our main concern in this article is to consider the whole structure of the lattice of the closed subspaces of a Hilbert space and investigate its relation to the concept of a quantale. We shall do this by looking at a more general situation. We shall see that the quantales introduced by Mulvey and Pelletier are a special case of an involution semigroup satisfying some extra properties. Girard (1989) raised the question of whether linear logic has something to do with quantum mechanics. In his original paper (Girard, 1987) he introduced a lattice which after the work of Yetter (1991) is called now a Girard quantale. We shall see that this kind of quantale is again a special case of an involution semigroup. More than 30 years ago, Foulis (1960) introduced the notion of a Baer $*$-semigroup and he proved that any orthomodular lattice can be viewed as a Baer $*$-semigroup.

After all these isolated results we feel it is time to unify all the concepts and introduce an algebraic structure which captures all these results. The paper is divided as follows. In Section 2 we introduce the notion of an orthomodular lattice and we show that the lattice of all endofunctors having a right adjoint forms a Baer $*$-semigroup. In Section 3 we look at the concept of a Girard quantale and we show how can we see this quantale as an involution semigroup. Section 4 deals with the concepts introduced by Mulvey and Pelletier; we shall see how they can generate an involution semigroup. We shall in fact draw attention to involution posets in the whole paper.

2. ORTHOCOMPLEMENTED LATTICES AND ORTHOMODULAR LATTICES

Instead of working only on the special case of the closed subspaces of a Hilbert space, we shall introduce the general notion of an orthomodular lattice; since we shall deal also with orthocomplemented lattices, we introduce both definitions.

Definition 2.1. Let L be an arbitrary lattice. We say L is an *orthocomplemented* lattice if there is a unary operation (denoted by $\perp$) of L satisfying the following properties:

1. $a^{\perp\perp} = a$
2. $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$
3. $a \vee a^{\perp} = 1$
4. $a \wedge a^{\perp} = 0$

for all $a, b \in L$. We shall say L is an *orthomodular* lattice if in addition it *satisfies the orthomodular law*: Given two arbitrary elements $a, b \in L$ such that $a \leq b$ the following identity holds:

$$b = a \vee (a^{\perp} \wedge b)$$

Clearly, the obvious example is the lattice of closed subspaces of a Hilbert space H [we shall denote this lattice by $\mathscr{C}(H)$ in the rest of this paper]; see, for instance, Kalmbach (1983) for a proof of this fact. The last identity is called the orthomodular law instead of the modular law, because the only Hilbert spaces satisfying the modular law are those having finite dimension (as vector spaces); see, for instance Halmos (1957) for a proof of this fact.

$\mathscr{C}(H)$ is not only a lattice, it is a complete lattice. Therefore we have the concepts of complete orthocomplemented lattices and complete orthomodular lattices, where condition 2 is replaced by $(\bigvee_{i \in I} a_i)^{\perp} = \bigwedge_{i \in I} a_i^{\perp}$. We are now ready to introduce the main operation of this lattice. This operation was first introduced by Finch (1970) in a different context. Suppose L is an arbitrary ortholattice and a is an arbitrary element of L; then we define the following endomorphism of L, denoted by $\phi_a: L \to L$:

$$\phi_a(b) = (b \vee a^{\perp}) \wedge a$$

for any element b in L. In Román and Rumbos (1988, 1991, n.d.-*a*) the rhs of the last identity was introduced as a binary operation of L, but we shall see it is better to consider this as an arrow. In Kalmbach (1983) there is a characterization of when an ortholattice is in fact an orthomodular lattice in terms of ϕ_a. For the sake of completeness we state this as a lemma:

Lemma 2.2. Let L be an arbitrary ortholattice. A necessary and sufficient condition for L to be an orthomodular lattice is the following condition:

$$\forall b \in L, \qquad b \leq a \;\Rightarrow\; \phi_a(b) = b$$

for all elements a in L.

For the rest of this section we shall assume L to be an orthomodular lattice. We now list some properties of the arrow ϕ_a.

Proposition 2.3. The arrow $\phi_a: L \to L$ satisfies the following:

1. ϕ_a is idempotent.
2. ϕ_a has a right adjoint, namely $\psi_a: L \to L$ given by the rule

$$\psi_a(b) = (a \wedge b) \vee a^{\perp}$$

3. For any element b of L the following is true:

$$\phi_a((\phi_a(b^{\perp})^{\perp})) \leq b$$

Proof. See Román and Rumbos (1988) for a proof of the first two properties. We shall look at the third claim. An easy calculation shows that the lhs of condition 3 is equal to $(a \wedge b)^{\perp} \wedge a$, which clearly is less than or equal to a, using the fact $\phi_a(a^{\perp}) = 0$. ■

Remark. The arrow ψ_a defined in condition 2 is very important in quantum mechanics; it is called the *Sasaki hook.* Notice also we do not need to assume any completeness property of L to prove Proposition 2.3. By condition 2, we know that if L has arbitrary joins, then ϕ_a will preserve it. Condition 3 is very important; in fact this is the key property for the representation of any orthomodular lattice as a Baer $*$-semigroup. We begin first with some definitions.

Definition 2.4. By an involution semigroup we mean a semigroup S equipped with a unary operation $*: S \to S$ such that for all x, y elements of S, $(xy)^* = y^*x^*$. We shall say $x \in S$ is a projection if $x = x^*$. We denote by $P(S)$ the set of projections of S.

One natural question is, How can we see an involution poset P as an involution semigroup? The idea of doing this is to look at the lattice of all endomorphisms of P denoted by $R(P)$ having a right adjoint. In order for this paper to be self-contained, we introduce the concept of an involution poset.

Definition 2.5. By an involution poset P we mean a poset P together with a unary operation $*: P \to P$ such that for any elements a, b of P we have:

1. $a{*}{*} = a$.
2. If $a \leq b$, then $b* \leq a*$.

Suppose then we have an involution poset $(P, *)$; we are now ready to introduce the main definition of this paper, as we said before we want to see how we can think of an involution poset as an involution semigroup. We have then the following:

Definition 2.6. Let $(P, *)$ be an involution poset. Suppose $f, g: P \to P$ are two arbitrary endomorphisms of P, then we say f is *dual* to g iff the following inequalities hold:

$$f(g(a*)*) \leq a, \qquad g(f(a*)*) \leq a$$

for any element a of P.

The reader perhaps may ask for examples of this kind of arrow; the next result will give a nice way of getting examples. We state this as the following:

Proposition 2.7. Let P be an involution poset. Suppose $f, g: P \to P$ are two arbitrary endomorphisms of P; then the following conditions are equivalent:

1. f is dual to g.
2. f has a right adjoint

Proof. Suppose that $f: P \to P$ has a dual $g: P \to P$, consider the endomorphism of P, h, given by the following rule: if $a \in P$, then $h(a) = g(a*)*$. We will see that h is a right adjoint to f.

Indeed, suppose $f(a) \leq b$; then $b* \leq f(a)*$ and $g(b*) \leq g(f(a)*)$ and hence $g(f(a)*)* \leq g(b*)* = h(b)$, but $g(f(a)*) = g(f(a**)*) = \leq a*$, since g is dual to f and we have therefore $h(f(a)) \leq a** = a$; i.e., $a \leq h(b)$. conversely, if $a \leq h(b)$, then $f(a) \leq f(g(b*)*) \leq b$. Hence, we have shown f is left adjoint to h.

A similar argument shows that if f has a right adjoint h, then endomorphism g defined by $g(a) = h(a*)*$ (for $a \in P$) is dual to f. ■

Remark. Notice we do not need to assume any completeness property for the poset P. In fact, if we know that an arbitrary endomorphism f of P has a dual, then f will preserve all suprema existing in P. This result contrasts with the considerations made by Mulvey and Pelletier (1991); see also Section 4 of this paper.

We will apply now all these results to the concrete case of an arbitrary orthomodular lattice. First of all, suppose L is an arbitrary orthomodular lattice L; then clearly L is an involution poset; the involution is given by the unary operation $\perp: L \to L$, which we will denote in the rest of the section as $*$. Denote by $R(L)$ the set of all endomorphisms of L having a dual arrow. The reader might ask if there is a nontrivial example of such an endomorphism. We shall see that there are many elements of $R(L)$, in fact, as least as many elements as in L.

Lemma 2.8. Let L be an arbitrary orthomodular lattice. If a denotes an arbitrary element of L, then the arrow $\phi_a: L \to L$ is *self-dual.*

Proof. Indeed, by Proposition 2.3 we know that ϕ_n is idempotent and has a right adjoint, namely ψ_a. Now, by the last proposition we know that the arrow $g: L \to L$ defined by $g(b) = \psi_a(b*)*$, for any b in L, is a dual arrow for ϕ_a. An easy calculation shows that h is equal to ϕ_a. ■

In particular, there is natural morphism from L into $R(L)$, taking any element a of L to ϕ_a. Moreover, $R(L)$ is a Baer $*$-semigroup. A Baer $*$-semigroup (S, K) is an involution semigroup S together with a focal ideal K. This means K is a two-sided ideal and for every element x of S the set $\{y \in S | xy \in k\}$ is a principal ideal. In our case the focal ideal is $\{0\}$. We can state now the main result of this section, which is due to Foulis; see Foulis (1960) for details and comments.

Theorem 2.9. (Foulis). Let L be any orthomodular lattice; then $(R(L), \{0\})$ is a Baer $*$-semigroup and the correspondence $a \to \phi_a$ between L and $P(R(L))$ is an isomorphism preserving the orthocomplementation. ■

We shall call this the Foulis representation theorem. Notice that we do not need to assume L is a complete orthomodular lattice. We shall look now at the similarities between orthomodular lattices and Girard quantales.

3. GIRARD QUANTALES

Girard quantales were introduced by Yetter (1991), the main purpose apparently to generalize the construction made by Girard (1987). Later Rosenthal (1991) wrote a paper concerning Girard quantales; his results are contained in Rosenthal (1990), where the reader can find some comments and examples. Barr (1991) gave as an example of his general construction of getting models of linear logic the case when we take a complete lattice, which is precisely what Rosenthal did in his paper. Again for the sake of completeness let us introduce the concept of a Girard quantale. First of all, recall that a quantale Q is a complete lattice endowed with a binary operation &, which is not necessarily commutative, such that the operation & commutes with arbitrary suprema on each side; in particular, this operation; viewed as a functor, has two right adjoints, which are denoted in general by $\to_l$ and $\to_r$, where the subscripts denote which right adjoint we are taking. We have now the following:

Definition 3.1. By a Girard quantale Q we mean a quantale having a cyclic dualizing element d. That is to say, there is an element d of Q such that for any element a of Q the following identities hold:

1. $a \to_l d = a \to_r d$.
2. $(a \to d) \to d = a$.

The important thing about this element d of Q is that the operation $(-) \to d$ becomes an involution for Q in such a way that the following identities can be shown; see Rosenthal (1990) for a proof of these facts (we keep our notation for the involution): For any elements a, b of Q:

1. $(a \vee b)^* = a^* \wedge b^*$.
2. $(a \wedge b)^* = a^* \vee b^*$.
3. $a \to_l b = (a \,\&\, b^*)^*$.
4. $a \to_r b = (b^* \,\&\, a)^*$.
5. $1^* = 0$.
6. $0^* = 1$.

First of all, consider the similarities with the equations described for orthomodular lattices. These identities look complicated because we must remember which right adjoint we are taking. However, we can rethink these identities, taking again the lattice of all endofunctors of Q having a right adjoint. Denote this lattice by $R(Q)$ and consider the following elements of $R(Q)$: $\lambda_a(x) = a \,\&\, x$ and $\rho_a(x) = x \,\&\,$ a, where a is a fixed element of Q and x is an arbitrary element of Q; we have then the following:

Lemma 3.2. The functors λ_a, $\rho_a: Q \to Q$ are dual.

Proof. Of course we know that these functors have a dual by Proposition 2.7. An easy calculation shows that $\lambda_a(\rho_a(b^*)^*) \leq b$ and $\rho_a(\lambda_a(b^*)^*) \leq b$, using the identities described above.

We can apply then the results stated in the last section in order to get a Baer $*$-semigroup. The only thing we need is to consider a focal ideal of $R(Q)$. Notice again the crucial idea here is to consider the lattice of all endofunctors of Q having a right adjoint.

We shall look now at the case of Gelfand quantales and the relation with Baer $*$-semigroups.

4. GELFAND QUANTALES

The concepts we shall discuss here were introduced by Mulvey and Pelletier (1991). The title of the paper, "The quantization of the calculus of relations," suggests that they in fact look at a new and different approach to the subject of the lattice of closed subspaces of a Hilbert space. We want to point out that Lambek (n.d.) discusses carefully one of the motivating examples of Mulvey and Pelletier, namely the case of relations of a given set. We shall look here at the second example they studied in their paper. We introduce several technical definitions in order for the reader to understand all the concepts; we shall follow Mulvey and Pelletier's approach. First of all they introduce the notion of an *involutive quantale*; this is nothing but a quantale Q together with an involution $*: Q \to Q$; of course this unary operation must satisfy the following identities: For any pair of elements a and b of Q:

1. $(a^*)^* = a$.
2. $(a \,\&\, b)^* = b^* \,\&\, a^*$.

Here they make the distinction that a quantale has a unit for the operation & and the top element of Q. Hence if Q has a unit e they must assume $e^* = e$. Now, given any quantale Q and an element a of Q we say a is right-sided if for every element b of Q the following holds:

$$a \,\&\, b \leq a$$

Denote by $D(Q)$ the set of all right-sided elements of a quantale Q. We can now introduce the notion of a Gelfand quantale.

Definition 4.1. A unital quantale Q is a *Gelfand quantale* if Q is involutive and the right-sided elements of Q satisfy the following property:

$$a \mathbin{\&} a^* \mathbin{\&} a = a$$

In their paper, they discuss in detail the case when we take as a basic example the lattice of all endofunctors of $C(H)$ having a right adjoint. We know already how to generate a Baer $*$-semigroup from this lattice. One natural question is: What are the relations between all the concepts they introduced and the constructions we described in Section 2? Well, as we pointed out in the introduction, the big difference between the two approaches is that they consider only complete *orthocomplemented* lattices. Indeed, we recall now the basic constructions by Mulvey and Pelletier introduced in their paper. In order to do that, we need to start with a *complete orthocomplemented* lattice S and consider the lattice of all endofunctors of S having a right adjoint, which we denote by $R(S)$. Mulvey and Pelletier claim that $R(S)$ has an involution given by the following rule: if $\phi \in R(S)$, then ϕ^* is given by

$$\phi^*(s) = \Bigl(\bigvee_{\phi(t) \leq s^{\perp}} t \Bigr)^{\perp} \cdots \text{(I)}$$

where $\perp: S \to S$ denotes the orthocomplement of S. This definition looks complicated and in fact is difficult to handle. By Proposition 2.7, we know that if a functor has a right adjoint, then it has a dual. Hence there must be some relation between these two concepts. We shall see that they are actually *the same*, as shown in the next proposition:

Proposition 4.2. Let S be any complete orthocomplemented lattice and consider the lattice $R(S)$ of all endofunctors having a right adjoint. Then given any element ϕ of $R(S)$, the function defined by (I) is the dual of ϕ. In particular, $R(S)$ is nothing but the lattice of all endofunctors having a dual.

Proof. We must check the conditions of Definition 2.6. First of all, notice that if $s \in S$, then it is clear that $\phi((\phi^* s)^{\perp}) \leq s^{\perp}$, simply because $[\phi^*(s)]^{\perp} = \bigvee_{\phi(t) \leq s^{\perp}} t$. Now, $\phi^*((\phi^* s)^{\perp} \leq s^{\perp}$ holds since the lhs of the last inequality can be written as follows:

$$\phi^*((\phi s)^{\perp}) = \bigwedge_{\phi(s^{\perp})^{\perp} \leq \phi(t)^{\perp}} t^{\perp} \cdots \text{(II)}$$

Hence, ϕ^* is the dual of ϕ and the proof is complete. ■

We can say now what the real difference is between Mulvey and Pelletier's approach and the results stated in Section 2 of this paper. First of all they forgot to consider the following inequality: $\phi^*((\phi s)^{\perp}) \leq s^{\perp}$. As we pointed out in Section 2 the main advantage of considering the dual of an endofunctor is that you have a very simple way of constructing an involution semigroup. Moreover, when we are dealing with an *orthomodular lattice* it is even better, since we get a Baer $*$-semigroup for free and the original lattice can be embedded in a very natural way by taking the left adjoint of the Sasaki hook. Notice also that we never need any completeness property; for them this is crucial and this is the main reason they start with a complete orthocomplemented lattice.

As a corollary of all these results, we think the best way of studying all these kinds of lattices is by taking the general framework: the lattice of all endofunctors having a right adjoint which we know is equivalent to the lattice of all endofunctors having a dual. Also, the dual of a functor and the orthocomplement operation of the given lattice have a very close connection which we hope is now clear.

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This work started when I was visiting J. Lambek, and the idea of looking at involution posets was inspired by a lecture given by Lambek explaining the relation between linear logic and the lattice of relations of a set [see Lambek (n.d.) for more details]. In a series of papers (Román and Rumbos, 1988,1991, n.d.-*a*,*b*) we introduced a nonassociative binary operation in any orthomodular lattice, which in fact can be thought of as a model of nonassociative linear logic. It was not until I listened to the lecture by Lambek that I got the idea that all these concepts can be viewed as involution posets.

Finally, I want to express my sincere thanks to Jim Lambek for having invited him twice to McGill University and to F. W. Lawvere for many conversations. Part of this work was done when I was working at the National Autonomous University of Mexico and was a visitor at McGill University. Also, I want to express my sincere thanks to Mary Karmen for her support and help.

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